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TABLE OF CONTENTS

LIST OF TABLES AND FIGURES

MONITORING AN INPUT-OUTPUT MODEL FOR PRODUCTION. I. THE CONTROL CHARTS

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0. Abstract. Shewhart type control charts are ~~given~~ used for monitoring an input-output model against changes in form, against changes in its coefficients, and against changes in process variability. When a process is not in control due to changes in some coefficients, monitoring shifts to a diagnostic mode to identify the altered coefficients and thus the needed adjustments to production. Control limits from special aid tables are used; these are considered along with the choice of design. Operating characteristics of the charts are summarized under standard assumptions.

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Key words and phrases. Shewhart control charts, monitoring a model, monitoring the coefficients, diagnostic procedures, optimal operating conditions.

1. Introduction. Shewhart control charts are used routinely in monitoring the means and variances of quality characteristics in industrial production. These charts are easily maintained and interpreted and their operating characteristics are known. Other aspects of production are important in addition to the outgoing quality, yet little has been done towards monitoring them. This study deals specifically with the conversion of resources into units of product and with efficient resource use.

Production processes often may be formulated as input-output models of the type

$$y = \beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p + \varepsilon \qquad (1.1)$$

where y is the quantity produced; ε is a random error; $\{X_1, \ldots, X_p\}$ are the levels of inputs such as the number of machine hours used (X_1), the quantity of raw materials converted (X_2), and the number of man-hours of labor required (X_3); and $\{\beta_1, \ldots, \beta_p\}$ are rates of conversion of input to output such as machine productivity in units per hour (β_1), materials conversion in units of output per unit of raw materials used (β_2), and labor productivity in units per man-hour (β_3). The variables $\{X_1, \ldots, X_p\}$ frequently include levels of inputs and functions of these levels as in second-order response models.

In practice an input-output model often is established in a small-scale pilot plant or in a regular production facility under careful scrutiny by process engineers. Such model may or may not apply after the process has gone on line or has seen extensive use. Response surface methodology and evolutionary operation are concerned with finding levels

of the input variables to achieve the most efficient use of resources. In order that operating conditions thus determined remain optimal, it is presumed that both the form of the model and the values of its coefficients remain stationary through time. Both, however, may change. The form of the model may change to include variables earlier held constant or to reflect changes in the type of dependence on inputs; values of the coefficients may change due to wear in machines or declining worker morale, for example. In practice it may be possible to adjust the process to its original state by replacing worn parts or by improving labor relations in the examples cited. Or, if the model has changed irrevocably, it may be necessary to establish revised operating conditions taking into account the altered form.

The object here is to monitor a model against changes in form, against changes in its coefficients, and against changes in process variability. The study is arranged in two parts. Part I gives charts for monitoring these changes, and it supplies the basic properties of these charts under standard assumptions. Related topics as developed in Part II include (i) properties of the charts when the usual assumptions fail, (ii) the notion of drifting processes, and (iii) stochastic bounds for certain distributions under drifting. Special features are that different charts may be maintained on different schedules as needed and, in monitoring the coefficients, that a diagnostic mode is available to aid in identifying the altered coefficients. Various modifications of the basic procedures are given to allow for flexibility in their use.

2. Preliminaries. Consider an expanded model of the type

$$y = \alpha + \sum_{i=1}^{p} \beta_i X_i + \sum_{j=1}^{q} \gamma_j Z_j + \varepsilon \qquad (2.1)$$

where $\{X_1, \ldots, X_p\}$ are variables appearing in the original model and $\{Z_1, \ldots, Z_q\}$ are additional variables which may or may not be adjusted in determining operating conditions. This model encompasses several avenues as follows through which changes in the model (1.1) may occur.

One possibility is that $\{Z_1, \ldots, Z_q\}$ are further functions of the original input factors not included in $\{X_1, \ldots, X_p\}$; then (2.1) represents a structural change in the dependence of output on inputs. For example, (1.1) may be a second-order response function and (2.1) a function of third order. Operating conditions considered optimal often differ widely between two such models.

A second possibility arises when $\{Z_1, \ldots, Z_q\}$ are functions only of factors extraneous to the initial modeling in that they either were unknown or essentially were held fixed. Examples are batch temperature, line voltage, and the percent impurities of materials from a supplier, all of which may fluctuate appreciably when a process goes on line. When $\{Z_1, \ldots, Z_q\}$ are held fixed the models (1.1) and (2.1) are related through the expression

$$\beta_0 = \alpha + \sum_{j=1}^{q} \gamma_j Z_j \ . \tag{2.2}$$

While the level of output now depends on the values of extraneous variables, optimal levels of the original variables often do not under this version of the model (2.1).

As a third possibility combining the other two, $\{Z_1, \ldots, Z_q\}$ may be determined by levels of the original input factors and the levels of factors extraneous to these. This allows for structural changes in the original model as well as dependence on extraneous variables. When some of

$\{Z_1, \ldots, Z_q\}$ are functions of the levels of both types of factors, optimal levels of the original variables generally are altered even when $\{\beta_1, \ldots, \beta_p\}$ remain constant over time.

To monitor the model, observations $\{y_1, \ldots, y_n\}$ are generated on each sampling occasion at levels $\{\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_n\}$ specified according to a particular design, where $\underset{\sim}{X}_i = [X_{i0}, X_{i1}, \ldots, X_{ip}]$ and $X_{i0} \equiv 1$ for $i = 1, 2, \ldots, n$. In standard form the observational versions of (1.1) and (2.1) are

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{\varepsilon} \tag{2.3}$$

and

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{Z}\underset{\sim}{\gamma} + \underset{\sim}{\varepsilon} \tag{2.4}$$

respectively, where $\underset{\sim}{y} = [y_1, \ldots, y_n]'$; $\underset{\sim}{X}$ is a design matrix of order $[n\times(p+1)]$ and rank $p+1$; $\underset{\sim}{\beta} = [\beta_0, \beta_1, \ldots, \beta_p]'$ with β_0 replacing α in expression (2.1); $\underset{\sim}{Z}(n\times q)$ is a matrix consisting of levels of the additional variables, often unknown; $\underset{\sim}{\gamma} = [\gamma_1, \ldots, \gamma_q]'$; and $\underset{\sim}{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_n]'$ is a vector of errors, all mutually uncorrelated and having the same variance σ^2. Here the primes denote transposition. Usually the same experimental design is repeated on successive occasions, often with a single point replicated to give the observations $\{y_1^*, \ldots, y_m^*\}$ having the sample mean $\bar{y}^*$. On each sampling occasion we compute one or more of (i) the Gauss-Markov estimates $\hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{y}$ under the model (1.1); (ii) the residual sum of squares $R(\hat{\underset{\sim}{\beta}}) = (\underset{\sim}{y} - \underset{\sim}{X}\hat{\underset{\sim}{\beta}})'(\underset{\sim}{y} - \underset{\sim}{X}\hat{\underset{\sim}{\beta}})$; and (iii) the sample variance $s^2 = [(y_1^* - \bar{y}^*)^2 + \ldots + (y_m^* - \bar{y}^*)^2]/(m-1)$. The latter provides a reasonable estimate for σ^2 regardless of the underlying model. On occasion σ^2 is assumed to be known or, if stationary over time, to have been estimated using the sample variance s_0^2 from a start-up period based on ν degrees of freedom.

The ensuing developments are based on the following standard assumptions.

(i) Independent random outcomes are observed on successive sampling occasions.

(ii) The errors $\{e_1, \ldots, e_n\}$ are distributed normally.

(iii) The control values $\{\beta_0^*, \beta_1^*, \ldots, \beta_p^*, \sigma_0^2\}$ for parameters of the original model are known.

Accordingly, needed values from standard distributions are $F_\alpha(r,\nu)$, the $100(1-\alpha)$ percentile of the central F distribution having r and ν degrees of freedom, and $\chi_\alpha^2(\nu)$, the $100(1-\alpha)$ percentile of the central chi-squared distribution having ν degrees of freedom. Other special values are $t_\alpha(p,\nu,\underset{\sim}{R})$, the upper percentile of the central p-dimensional Student's distribution having ν degrees of freedom and the correlation matrix $\underset{\sim}{R}(p \times p)$ such that $P(t_1 \leq t_\alpha(p,\nu,\underset{\sim}{R}), \ldots, t_p \leq t_\alpha(p,\nu,\underset{\sim}{R})) = 1-\alpha$; and $F_\alpha(p,\nu,\underset{\sim}{R})$, the two-sided version such that $P(t_1^2 \leq F_\alpha(p,\nu,\underset{\sim}{R}), \ldots, t_p^2 \leq F_\alpha(p,\nu,\underset{\sim}{R})) = 1-\alpha$. These distributions are discussed in reference [4], for example; special aid tables are cited in a later section.

3. The Control Charts. To monitor a process samples are taken on successive occasions and appropriate statistics are computed. A control chart of Shewhart's type consists of successive values of a statistic plotted against time as the horizontal scale. Alpha-level control limits are provided beyond which the chart signals that the process is not in control, with α the probability of exceeding the limits on each occasion when the process is in control. A typical chart with one-sided limit is depicted in Figure 1a; a two-sided version is shown in Figure 1b. In the sections following we consider monitoring an input-output model against changes of

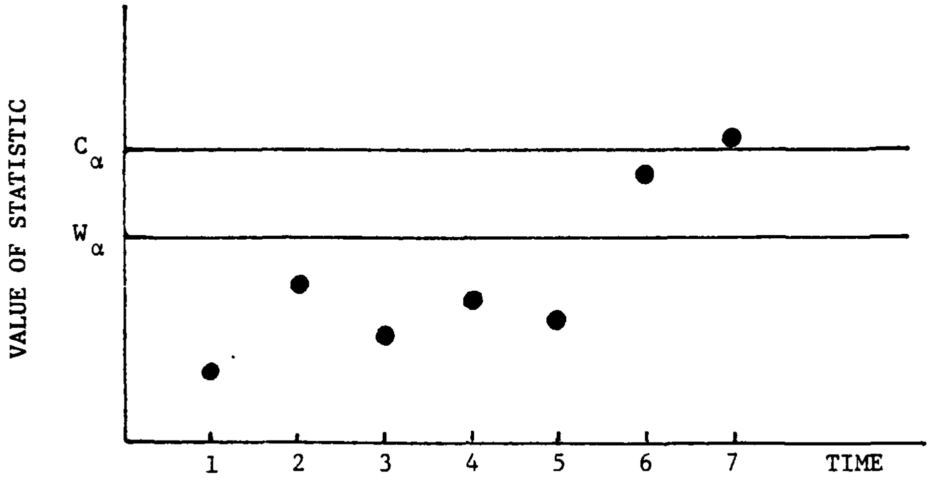

FIGURE 1a. A typical Shewhart control chart with warning line W_α, upper control limit C_α, and values warning at time 6 and signaling at time 7.

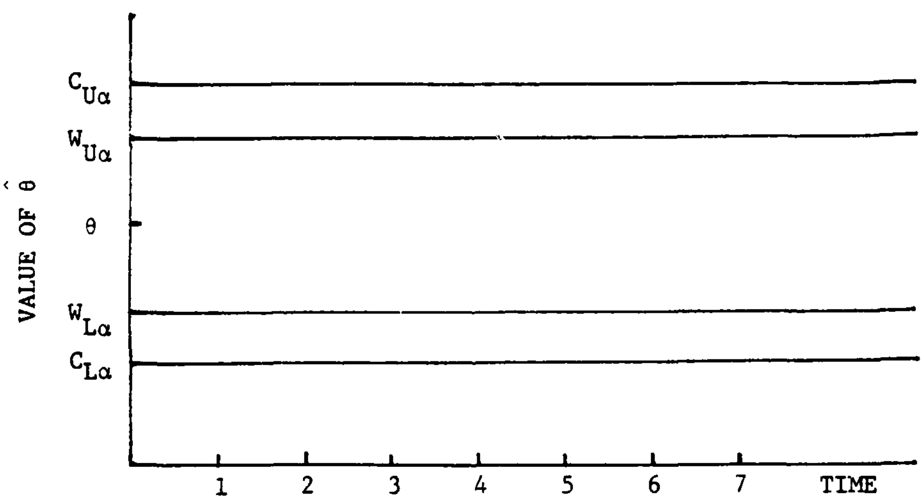

FIGURE 1b. A typical Shewhart control chart for plotting a statistic $\hat{\theta}$ with target value θ, lower ($W_{L\alpha}$) and upper ($W_{U\alpha}$) warning lines, and lower ($C_{L\alpha}$) and upper ($C_{U\alpha}$) control limits.

the types mentioned. In particular, for each type of change we stipulate (i) the statistics to be plotted, (ii) the control limits appropriate, and (iii) the corrective action to be taken when a chart signals.

3.1 Monitoring the structure. The purpose is to monitor the model (1.1) against changes of the form (2.1). The basic procedure uses an F chart of the type shown in Figure 1a with an upper control limit.

THE CHART. Plot values of the statistic

$$F_1 = R(\hat{\underset{\sim}{\beta}})/(n-p-1)s^2. \tag{3.1}$$

THE CONTROL LIMIT. The chart signals a change in structure whenever

$$F_1 > F_\alpha(n-p-1,\ m-1). \tag{3.2}$$

CORRECTIVE ACTION. Use residual analysis to identify changes in the form of the model and to isolate extraneous variables now pertinent. Determine whether these extraneous variables may be adjusted for efficiency. If necessary revise the model and determine revised operating conditions.

OPTIONAL PROCEDURES. If the process variance σ^2 is known the statistic $X_1^2 = R(\hat{\underset{\sim}{\beta}})/\sigma^2$ may be plotted on a chi-squared chart similar to Figure 1a with $\chi_\alpha^2(n-p-1)$ as the upper control limit. If σ^2 is stationary over time but otherwise unknown, the statistic $F_1' = R(\hat{\underset{\sim}{\beta}})/(n-p-1)s_0^2$ may be plotted on an F chart and $F_\alpha(n-p-1,\ \nu)$ used as a control limit, where s_0^2 is an initial estimate based on ν degrees of freedom.

3.2 Monitoring the coefficients. The purpose is to monitor against general alternatives changes in the coefficients of the model (1.1) from their designated values $\underset{\sim}{\beta}^* = [\beta_0^*,\ \beta_1^*,\ \dots\ ,\beta_p^*]'$. The basic procedure uses an F chart as in Figure 1a.

THE CHART. Plot values of the statistic

$$F_2 = S(\hat{\beta},\beta^*)/(p+1)s^2 \tag{3.3}$$

where $S(\hat{\beta},\beta^*) = (\hat{\beta}-\beta^*)'X'X(\hat{\beta}-\beta^*)$.

THE CONTROL LIMIT. The chart signals a change in some one or more coefficients whenever

$$F_2 > F_\alpha(p+1,\ m-1). \tag{3.4}$$

CORRECTIVE ACTION. Monitoring shifts to the diagnostic mode of Section 3.3 in order to identify the altered coefficients and thus the specific adjustments to be made to the process.

OPTIONAL PROCEDURES. Two types of options are considered.

(i) If σ^2 is known the statistic $X_2^2 = S(\hat{\beta},\beta^*)/\sigma^2$ may be plotted on a chi-squared chart and $\chi_\alpha^2(p+1)$ used as a control limit. If σ^2 is stationary over time, justifying use of an initial estimate s_0^2 having ν degrees of freedom, the statistic $F_2' = S(\hat{\beta},\beta^*)/(p+1)s_0^2$ may be plotted on an F chart and $F_\alpha(p+1,\ \nu)$ used as a control limit.

(ii) Suppose some subset $\{X_{i_1}, \ldots ,X_{i_t}\}$ of the p variables $\{X_1, \ldots , X_p\}$ may be adjusted in determining efficient operating conditions, and let $\xi = [\beta_{i_1}, \ldots ,\beta_{i_t}]'$ be the corresponding coefficients. Then it may be desired to determine whether designated operating conditions for these variables remain optimal, i.e., whether the coefficients ξ all achieve their designated values $\xi^* = [\beta^*_{i_1}, \ldots ,\beta^*_{i_t}]'$. This may be accomplished on plotting the statistic $F_2^* = S(\hat{\xi},\xi^*)/ts^2$ on an F chart using $F_\alpha(t,m-1)$ as a control limit, where now $S(\hat{\xi},\xi^*) = (\hat{\xi}-\xi^*)'M^{-1}(\hat{\xi}-\xi^*)$ and M is the $(t\times t)$ matrix obtained from $(X'X)^{-1}$ on deleting rows and columns not in the set $\{i_1, \ldots ,i_t\}$ and rearranging those remaining in the order $(i_1, \ldots ,i_t)$. In particular, when

optimal conditions depend on the levels of $\{X_1, \ldots, X_p\}$ but not on the overall level of response, this gives a procedure for monitoring $\{\beta_1, \ldots, \beta_p\}$ ignoring the value of β_0.

3.3 The diagnostic mode. The purpose is to determine which coefficients have changed and thus the specific adjustments to be made to the process such as repairing machines or improving worker productivity. The basic procedure gives multipurpose use to a single control chart of the type shown in Figure 1a; other options are noted. Write $(\underset{\sim}{X}'\underset{\sim}{X})^{-1} = \underset{\sim}{M} = [m_{ij}]$ with i and j ranging from 0 through p, and let $\underset{\sim}{R} = [\rho_{ij}]$ be the correlation matrix for $\hat{\underset{\sim}{\beta}}$, which is determined by the choice of design.

THE CHART. Using different symbols plot the several statistics

$$t_i^2 = (\hat{\beta}_i - \beta_i^*)^2/m_{ii}s^2 \ , \ i = 0, 1, \ldots, p \tag{3.5}$$

on a single chart as in Figure 1a.

THE CONTROL LIMIT. The coefficient β_i is declared significantly different from its control value β_i^* whenever

$$t_i^2 > F_\alpha(p+1, m-1, \underset{\sim}{R}) \tag{3.6}$$

for each $i = 0, 1, \ldots, p$.

CORRECTIVE ACTION. Adjustments of the appropriate types are initiated for each coefficient found to differ significantly from its control value.

OPTIONAL PROCEDURES. Four options are considered.

(i) The estimate s^2 may be replaced by σ^2 if known or by s_0^2 from an initial period if variance is stationary. The corresponding changes in the control limit are to replace m-1 in $F_\alpha(p+1, m-1, \underset{\sim}{R})$ by ∞ and ν respectively for the two cases.

(ii) Separate charts may be maintained for the several coefficients as in Figure 1b. Here the target value is β_i^* for the ith chart, and the upper and lower control limits are given by $\beta_i^* \pm [m_{ii}s^2 F_\alpha(p+1, m-1, \underset{\sim}{R})]^{\frac{1}{2}}$. This has the advantages that $\hat{\beta}_i$ may be plotted on its natural scale rather than the standardized scale of the single chart, and that trends above as well as below the control value may be followed. The disadvantage is that several charts are required.

(iii) Because the F chart of Section 3.2 monitors against general alternatives, it is appropriate that the diagnostic mode be carried out against two-sided alternatives. However, if one-sided alternatives are mandated throughout, then monitoring as in Section 3.2 may be omitted in favor of the following procedure. To monitor against one-sided upper alternatives, use different symbols to plot the statistics

$$t_i = (\hat{\beta}_i - \beta_i^*)/(m_{ii}s^2)^{\frac{1}{2}};\ i = 0, 1, \ldots ,p \tag{3.7}$$

on a single chart as in Figure 1a with $t_\alpha(p+1, m-1, \underset{\sim}{R})$ as the upper control limit. Alternatively, a separate chart may be maintained for each coefficient in which $\hat{\beta}_i$ is plotted on its natural scale with $\beta_i^* + (m_{ii}s^2)^{\frac{1}{2}}\ t_\alpha(p+1, m-1, \underset{\sim}{R})$ as an upper control limit. Against one-sided lower alternatives the foregoing procedures may be followed using $-t_\alpha(p+1, m-1, \underset{\sim}{R})$ and $\beta_i^* - (m_{ii}s^2)^{\frac{1}{2}}\ t_\alpha(p+1, m-1, \underset{\sim}{R})$ respectively as lower control limits.

(iv) The foregoing procedures including options all may be adapted for monitoring a subset $\underset{\sim}{\xi} = [\beta_{i_1}, \ldots ,\beta_{i_t}]'$ of the parameters against their control values $\underset{\sim}{\xi}^* = [\beta_{i_1}^*, \ldots ,\beta_{i_t}^*]'$. The essential changes are to replace p+1 by t and $\underset{\sim}{M}$ by the (t×t) matrix obtained on deleting rows and columns of $(\underset{\sim}{X}'\underset{\sim}{X})^{-1}$ not in the set $\{i_1, \ldots ,i_t\}$ and rearranging those remaining in the order $(i_1, \ldots ,i_t)$.

3.4 Monitoring the process variance. The purpose is to determine whether the process variance σ^2 is maintained at its control value σ_0^2. Adjustment is initiated on evidence that σ^2 exceeds σ_0^2.

THE CHART. Plot the statistic

$$X_3^2 = (m-1)s^2/\sigma_0^2 \tag{3.8}$$

on a chi-squared chart of the type shown in Figure 1a.

THE CONTROL LIMIT. The variance of the process is inferred to be excessive whenever

$$X_3^2 > \chi_\alpha^2(m-1). \tag{3.9}$$

CORRECTIVE ACTION. Adjustments to reduce the process variability are undertaken when the chart signals.

OPTIONAL PROCEDURES. Often the control variance σ_0^2 is not assumed known, but instead an estimate s_0^2 based on ν degrees of freedom is used from an initial period when the process is known to be in control. Then the statistic $F_3 = s^2/s_0^2$ is plotted on an F chart as in Figure 1a with $F_\alpha(m-1, \nu)$ as a control limit.

4. Special Aid Tables. Standard tables are widely available for use with the F and chi-squared charts irrespective of the particular experimental design. The same control limit applies on successive occasions as long as n is held fixed, whether or not the designs are all identical.

Owing to an excess of parameters, tables of percentiles associated with multidimensional Student's distributions are available only for the equicorrelated case in which $\rho_{ij} = \rho$ for all $i \neq j$. A summary of these sources is given in the accompanying Table 1 together with ranges for the

TABLE 1. Special aid tables related to p-dimensional Student's distributions having ν degrees of freedom and the common correlation parameter ρ.

Percentiles Tabulated	Range of Values				Reference
	α	p	ν	ρ	
$t_\alpha(p,\nu,\rho)$	*	1(1)10	5(1)35	0(0.1)0.9	[5]
$t_\alpha(p,\nu,\rho)$	* *	1(1)10	5(1)35	0(0.1)0.9	[7]
$F_\alpha(p,\nu,0)$	*	1(1)12	5(1)45	0	[1]
$F_\alpha(p,\nu,\rho)$	*	1(1)10	5(1)35	0.1(0.1)0.9	[6]

* α = 0.01, 0.025, 0.05, 0.10   * * α = 0.01, 0.05

several parameters in question. Here $t_\alpha(p,\nu,\rho)$ and $F_\alpha(p,\nu,\rho)$ are upper percentiles of the equicorrelated distributions having the correlation parameter ρ; these correspond respectively to $t_\alpha(p,\nu,\underset{\sim}{R})$ and $F_\alpha(p,\nu,\underset{\sim}{R})$ in the general case. In addition, tables of $1-\alpha = P(t_1 \le a,\ t_2 \le a)$ are given in reference [8], and values of $1-\alpha = P(|t_1| \le a,\ |t_2| \le a)$ in reference [9], in the bivariate case for the values $a = 1.0(0.1)5.5$, $\nu = 5(1)35$, and $|\rho| = 0(0.1)0.9$. Tables of $1-\alpha = P(|t_1| \le a, \ldots, |t_p| \le a)$ are given in reference [2] for $a = 0.2(0.2)6.0$; $\nu = 4(2)12(4)24, 30, \infty$; $p = 2(2)20$; and $\rho = 0(0.1)0.9$ and $\rho = -1/(p-1)$, all based on Monte Carlo methods. Abbreviated tables of $t_\alpha(p,\nu,\rho)$ and $F_\alpha(p,\nu,\rho)$ appear in Tables 2 and 3 as taken from the sources cited in Table 1. Values of $t_\alpha(p,\nu,0)$ and $F_\alpha(p,\nu,0)$ assume a special role for reasons to be given.

On occasion the design matrix $\underset{\sim}{X}$ may be chosen arbitrarily. In order that the same control limits apply in the diagnostic mode, it is necessary that successive designs have the same correlation parameters. To expedite the diagnostic mode it is recommended whenever possible that $\underset{\sim}{X}$ be chosen such that $\rho_{ij} = \rho$ for all $i \neq j$, for then the special aid tables apply directly. For this it suffices that $(\underset{\sim}{X}'\underset{\sim}{X})^{-1} = \underset{\sim}{D} + \rho\underset{\sim}{a}\underset{\sim}{a}'$ with $\underset{\sim}{D}$ diagonal and $\underset{\sim}{a}$ chosen suitably or, equivalently, that $(\underset{\sim}{X}'\underset{\sim}{X}) = \underset{\sim}{D}^{-1} + \gamma\underset{\sim}{b}\underset{\sim}{b}'$ have the same structure (cf. reference [3], page 170). In particular, an orthogonal design yields $\rho_{ij} = 0$ for all $i \neq j$, in which case tables of $t_\alpha(p,\nu,0)$ and $F_\alpha(p,\nu,0)$ are appropriate. Unfortunately, equicorrelated designs are not possible in many cases such as second order response models. However, by using appropriate probability inequalities the tabulated percentiles often may be applied conservatively to more general correlation structures.

For one-sided Gaussian limits Slepian (1962) has shown that the probability $P_R(Z_1 \le a_1, \ldots, Z_p \le a_p)$ is an increasing function of each ρ_{ij},

TABLE 2. Selected values of $t_\alpha(p,\nu,\rho)$ from reference [5].

ν \ p	$\rho = 0$ 2	4	6	$\rho = 0.4$ 2	4	6	$\rho = 0.8$ 2	4	6
				$\alpha = 0.05$					
6	2.42	2.89	3.17	2.36	2.76	2.99	2.22	2.48	2.61
8	2.28	2.70	2.95	2.24	2.60	2.80	2.12	2.35	2.47
10	2.21	2.60	2.82	2.17	2.50	2.69	2.06	2.27	2.39
15	2.12	2.47	2.66	2.08	2.39	2.56	1.98	2.18	2.29
20	2.08	2.40	2.59	2.04	2.33	2.50	1.95	2.14	2.25
25	2.05	2.37	2.55	2.02	2.30	2.46	1.92	2.12	2.22
30	2.03	2.34	2.52	2.00	2.28	2.44	1.91	2.10	2.20
				$\alpha = 0.01$					
6	3.68	4.25	4.59	3.63	4.12	4.40	3.48	3.80	3.97
8	3.34	3.80	4.07	3.30	3.71	3.94	3.19	3.45	3.60
10	3.16	3.56	3.79	3.13	3.49	3.69	3.03	3.27	3.40
15	2.94	3.28	3.47	2.92	3.22	3.40	2.83	3.05	3.16
20	2.84	3.14	3.32	2.82	3.10	3.26	2.75	2.94	3.05
25	2.78	3.07	3.24	2.77	3.03	3.18	2.70	2.89	2.99
30	2.75	3.02	3.18	2.73	2.99	3.14	2.66	2.85	2.95

TABLE 3. Selected values of $F_\alpha(p,\nu,\rho)$ from references [1] and [6].

	$\rho = 0$			$\rho = 0.4$			$\rho = 0.8$		
ν \ p	2	4	6	2	4	6	2	4	6
				$\alpha = 0.05$					
6	8.50	11.48	13.43	8.31	10.95	12.61	7.56	9.16	10.09
8	7.39	9.78	11.32	7.24	9.37	10.70	6.63	7.95	8.71
10	6.81	8.90	10.24	6.68	8.55	9.72	6.15	7.33	8.00
15	6.12	7.87	8.96	6.02	7.60	8.56	5.58	6.59	7.17
20	5.81	7.41	8.40	5.72	7.17	8.05	5.32	6.26	6.80
25	5.64	7.14	8.07	5.55	6.93	7.76	5.17	6.07	6.58
30	5.52	6.98	7.87	5.44	6.77	7.57	5.07	5.95	6.45
				$\alpha = 0.01$					
6	18.24	23.58	27.07	17.94	22.70	25.72	16.67	19.63	21.35
8	14.51	18.26	20.68	14.31	17.70	19.82	13.44	15.61	16.87
10	12.72	15.75	17.68	12.57	15.33	17.05	11.87	13.69	14.73
15	10.75	13.02	14.44	10.65	12.75	14.04	10.14	11.57	12.39
20	9.92	11.87	13.08	9.83	11.67	12.78	9.40	10.67	11.39
25	9.45	11.25	12.35	9.38	11.07	12.09	8.99	10.17	10.85
30	9.16	10.85	11.89	9.10	10.70	11.66	8.73	9.86	10.50

where $\underset{\sim}{Z} = [Z_1, \ldots, Z_p]'$ is a p-dimensional standard Gaussian vector having the correlation matrix $\underset{\sim}{R} = [\rho_{ij}]$. Applying Slepian's result conditionally on s^2 and noting that inequalities are preserved on averaging over the conditioning variable, we have the following useful result. Let $\rho^* = \min\{\rho_{ij};\ i \neq j\}$; then the inequality

$$P(t_1 \leq t_\alpha(p,\nu,\rho^*), \ldots, t_p \leq t_\alpha(p,\nu,\rho^*)) \geq 1-\alpha \tag{4.1}$$

holds uniformly for all nonsingular correlation matrices $\underset{\sim}{R}$ having $\rho^* = \min\{\rho_{ij};\ i \neq j\}$, with equality when $\rho_{ij} = \rho^*$ for all $i \neq j$. Through this mechanism special aid tables now available can be applied conservatively for any design matrix $\underset{\sim}{X}$ in the sense that the actual operating level is no greater than α.

For two-sided limits a result of Šidák (1967) applies directly to give

$$P(t_1^2 \leq F_\alpha(p,\nu,0), \ldots, t_p^2 \leq F_\alpha(p,\nu,0)) \geq 1-\alpha \tag{4.2}$$

uniformly for every correlation matrix $\underset{\sim}{R}$, with equality when $\rho_{ij} = 0$ for all $i \neq j$ as in the case of orthogonal designs. Tables of $F_\alpha(p,\nu,0)$ as given in reference [1] thus apply conservatively in the case of two-sided limits for every choice of design.

5. Properties of the Charts. We examine performance characteristics of the charts under standard assumptions as set forth in Section 2. The run length of a chart is the number of successive samples taken before the chart signals that the process is not in control. For many purposes the operating characteristics of a chart may be summarized succinctly in terms of the distribution of its run lengths. We follow this approach here.

Consider first those F and chi-squared charts of the preceding sections not using s_0^2, and suppose the parameters monitored remain fixed through time. Then by independence and the identical distributions of successive statistics associated with each chart, it follows that its run-length distribution is geometric with parameter γ equal to the probability of exceeding the control limit on each occasion. A consequence is that each chart eventually signals with unit probability. Moreover, because for each chart the normal-theory tests have power increasing monotonically with departures from control, and because geometric distributions are stochastically decreasing in the parameter γ, it follows that the run lengths become stochastically smaller, and thus each chart tends to signal more frequently, with increasing departures from control.

Modifications to the basic charts were given using an estimate s_0^2 from an initial period. The resulting run-length distributions are complicated, being no longer geometric owing to dependence among the successive statistics. Moreover, even under independence the geometric property fails when the parameters monitored are not constant through time. Properties of the actual run-length distributions resulting from failure of these and other assumptions are considered in detail in Part II of this study.

Similar assessments may be given for the diagnostic charts. Here it is useful to define the run length as the number of sampling occasions in which no chart signals and thus no diagnoses are rendered. For example, the run-length distribution is geometric with parameter α when the process is in control and exact tables are used. If instead approximate procedures are used based on (4.1) or (4.2), then the actual run length is stochastically larger than one having the geometric distribution with parameter α, which thus provides a bound.

6. An Application. Taraman and Lambert [12] reported a study on conditions for machining steel in which the logarithm (y) of tool life in minutes was regressed on coded variables x_1, x_2, and x_3 representing logarithms of cutting speed, rate of feed, and depth of cut, respectively. A partially replicated central composite design was used. A second-order model of the type

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2 + \varepsilon \tag{6.1}$$

having only linear and quadratic terms in each variable was established provisionally. We treat this data set as one from a succession of experiments to be used for monitoring tool life in production. Taraman and Lambert [12] used the shapes of this and two related response surfaces in determining operating conditions. The shape of response thus is to be monitored in addition to adequacy of the model and its residual variance. As the shape of a second-order response surface is determined apart from centering by its second-order coefficients, we monitor the latter using one of the optional procedures of Section 3.2.

The design is central composite, built around a single replication of a 2^3 factorial experiment with vertices at ± 1 for the coded variables. The center point was replicated four times, and two replications were made at each of the coded values $\pm\sqrt{2}$ on the three coordinate axes. Estimates for parameters of the model (6.1) are given in Table 4 along with the values v_{ii} from the diagonal of $\underset{\sim}{V} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}$. The analysis of variance is summarized in Table 5, the pure error component giving the estimate $s^2 = 0.009275$ for σ^2 having 9 degrees of freedom.

To monitor for changes in the structure of the model (6.1) we compute the statistic $F_1 = 0.023171/0.009275 = 2.498$ as in Section 3.1. The upper control limit at the level $\alpha = 0.05$ is $F_{0.05}(8,9) = 3.23$, supporting the conclusion that the form of the model is as specified.

TABLE 4. Estimated coefficients for the model (6.1) and values $\{v_{ii}\}$ from the diagonal of $\underset{\sim}{V} = (\underset{\sim}{X}'\underset{\sim}{X})^{-1}$.

Parameter	Estimate	v_{ii}
β_0	3.500933	0.208330
β_1	-0.303106	0.062501
β_2	-0.092222	0.062501
β_3	-0.091540	0.062501
β_{11}	0.048230	0.083335
β_{22}	0.041642	0.083335
β_{33}	0.068205	0.083335

TABLE 5. Analysis of variance of the data in reference [12].

Source	Degrees of Freedom	Sum of Squares	Mean Square
Model	6	1.812867	0.302144
Lack of Fit	8	0.185366	0.023171
Error	9	0.083479	0.009275

In order to monitor the shape coefficients, we suppose that their target values from a base line period are $\xi^* = (\beta^*_{11}, \beta^*_{22}, \beta^*_{33})' = (0.13, 0.09, 0.03)'$. From the matrix $(X'X)^{-1}$ we extract the lower principal diagonal block of order (3×3) to get

$$M = \begin{bmatrix} 0.083335 & 0.020833 & 0.020833 \\ 0.020833 & 0.083335 & 0.020833 \\ 0.020833 & 0.020833 & 0.083335 \end{bmatrix} \qquad (6.2)$$

using the notation of Section 3.2 under option (ii). On evaluating $S(\hat{\xi}, \xi^*)$ as the quadratic form

$$(\hat{\xi} - \xi^*)'M^{-1}(\hat{\xi} - \xi^*) = 0.119455 \qquad (6.3)$$

we compute $F^*_2 = S(\hat{\xi}, \xi^*)/3s^2 = 4.293$. As the upper control limit at the level $\alpha = 0.05$ is $F_{0.05}(3,9) = 3.86$, the chart signals a shift in one or more shape coefficients from their target values. Monitoring now shifts to the diagnostic mode of Section 3.3 to identify which coefficients may have changed.

Proceeding as in Section 3.3 we compute the Studentized statistics $t_1^2 = (\hat{\beta}_{11} - \beta^*_{11})^2/m_{11}s^2 = 8.651$, $t_2^2 = (\hat{\beta}_{22} - \beta^*_{22})^2/m_{22}s^2 = 3.025$, and $t_3^2 = (\hat{\beta}_{33} - \beta^*_{33})^2/m_{33}s^2 = 1.888$, where from (6.2) $m_{11} = m_{22} = m_{33} = 0.083335$. It is clear from the form of M that $(\hat{\beta}_{11}, \hat{\beta}_{22}, \hat{\beta}_{33})$ are equicorrelated having the common correlation $\rho = 0.25$. The upper control limit required for the diagnostic chart thus is $F_\alpha(p,\nu,\rho)$ in the notation of Section 4 with $p = 3$, $\nu = 9$ and $\rho = 0.25$. Interpolating linearly between $F_{0.05}(3,9,0.2) = 8.26$ and $F_{0.05}(3,9,0.3) = 8.18$ using the tables in [6], we obtain the required critical value $F_{0.05}(3,9,0.25) = 8.22$. As t_1^2 exceeds this value and t_2^2 and t_3^2 do not, we infer that the second-order coefficient for cutting speed is significantly smaller than its target value. Thus either an adjustment to the process should be made, or operating conditions should be modified to reflect the altered coefficient.

If the process variance is to be monitored against the control value $\sigma_0^2 = 0.01$, we find that $X_3^2 = 9s^2/\sigma_0^2 = 8.348$. The upper control limit is $\chi_\alpha^2(\nu) = 16.919$ with $\alpha = 0.05$ and $\nu = 9$, indicating that the process variance remains in control.

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Shewhart type control charts are given for monitoring an input-output model against changes in form, against changes in its coefficients, and against changes in process variability. When a process is not in control due to changes in some coefficients, monitoring shifts to a diagnostic mode to identify the altered coefficients and thus the needed adjustments to

20.
production. Control limits from special aid tables are used; these are considered along with the choice of design. Operating characteristics of the charts are summarized under standard assumptions.